Counting Handout

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1 Introduction

Combinatorics (counting) is one of the most interesting fields in mathematics. Put simply, it is the study of counting things that are vast or complex. For instance, how many possible license plates are there if each license plate has 3 letters followed by 4 numbers? This question might seem daunting, but it's actually quite simple. After completing this handout, you'll be amazed by what you're able to count.

Theorem 1 (Fundamental Principle of Counting)**.** If there are p ways to do one thing and q ways to do another thing, there are pq ways to do those things together.

Example 1: Robert has 3 sweaters colored yellow, red and blue, and 4 hats colored green, indigo, orange, and white. How many sweater-hat pairs can Robert make?

Solution: According to the Fundamental Principle of Counting, there are $3 \cdot 4 = \boxed{12}$ ways to create sweater-hat pairs. This should make sense because for each sweater there are 4 hats it can be paired with. Because there are 3 sweaters, our calculation is $3 \cdot 4$.

2 Factorial Notation

Definition 1. The notation $n!$ denotes the product of all the integers from 1 to n. That is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Example 2: We can compute some simple factorials: $2! = (1)(2) = 2$, $3! = (1)(2)(3) = 6$, $4! = (1)(2)(3)(4) = 24$, and $5! = (1)(2)(3)(4)(5) = 120$.

Factorial notation isn't just for integers. To better understand how factorials work, we can consider an expression involving variables.

Example 3: Simplify the expression

$$\frac{(x+4)!}{(x+1)!}$$

Solution: Recall that $n!$ is the product of n consecutive integers, beginning at 1. We can write $(x+4)! = (x+4)(x+3)(x+2)(x+1)!$. Then our expression becomes

$$\frac{(x+4)(x+3)(x+2)(x+1)!}{(x+1)!} = (x+4)(x+3)(x+2)$$

since we can cancel the $(x+1)!$ in the numerator and denominator.

Note that by convention $0! = 1$. This makes sense because there is 1 way to arrange 0 things.

3 Permutations

Definition 2. An ordered arrangement of some set of elements.

Example 4: TAMH is a permutation of the letters of the word MATH.

Example 5: 5 people are standing in a line. How many ways are there to arrange the people in this line?

Solution: We can use the Fundamental Principle of Counting. There are 5 people who could be at the front of the line. After this first person is chosen, there are 4 people to be second in line. Now 2 people have been chosen, so there are 3 choices for who can be third. We can continue this until there is only one choice for the last person in line. This comes to $5! = \boxed{120}$.

Suppose instead we have more objects to choose from than the number of objects that we want to arrange. We still utilize the Fundamental Principle of Counting. However, we don't get an answer in the simple form $n!$.

Example 6: The National Honor Society wants to elect members to be President, Vice-President, and Treasurer. If there are 45 people in the organization, how many different ways are there to choose members to hold these positions?

Solution: There are 45 choices for who is President. After choosing the President, there are 44 choices for the Vice-President. Similarly, there are 43 choices for the Treasurer. According to Theorem 1, there are $45 \cdot 44 \cdot 43 = \boxed{85,140}$ ways to choose the positions.

Notice that our answer is equivalent to

$$\frac{45!}{(45-3)!} = \frac{45!}{42!} = (45)(44)(43)$$

This reveals an interesting formula that can be used to evaluate permutations.

Theorem 2 (Permutation Formula)**.** Let $P(n,r)$ be the number of ways to r items when choosing from a group of n items. Then

$$P(n,r) = \frac{n!}{(n-r)!}$$

It's important not to memorize this formula. Instead, understand why it works. Why does this formula calculate the number of ways to arrange r items when choosing from a group of n items?

Example 7: Isabelle is lining her teddy bears along the side of her bed. If she has 11 teddy bears and wants to display 5 on her bed, how many ways can she do this?

Solution: We arrange 5 items while choosing from a group of 11 items. The number of ways to do this is

$$\frac{11!}{(11-5)!} = \frac{11!}{6!} = (11)(10)(9)(8)(7) = 50,400$$

4 Overcounting

Overcounting is when you attempt to count something but the expression you have has items or events that you don't want. For example, if I want to find the number of arrangement of the letters in EVEN, I might first try 4!. However, this assumes that all the letters are distinct, even though we have two E's. In fact, we're overcounting by a factor of 2, but we'll discuss later why this is the case.

Overcounting isn't a bad thing. Often in problems, we might start with an expression that overcounts cases. Then we strategically subtract or divide out cases to get the correct answer.

Considering the number of ways to arrange letters in certain words is a great way to understand overcounting. When all the letters are distinct, like in RAMEN, a simple factorial expression can be used. However, what if there are duplicate letters?

For the word EVEN, we might first think there are 4! arrangements. Let's write out all the possible arrangements: EEVN, EENV, EVEN, ENEV, EVNE, ENVE, NEEV, NEVE, NVEE, VEEN, VENE, VNEE. There are 12, which is half of $4! = 24$. The actual count is half of 4! because each true arrangement is counted twice in 4!. For instance, EEVN is counted twice in 4! since the two E's are considered distinct.

What if there are more than two duplicate letters? Consider the word BELIEVE. There are 7 letters and 7! overcounts the number of arrangements. This is because 7! assumes that the three E's are distinct, like we're arranging $BE_1LIE_2VE_3$. Similar to the example with EVEN, each true arrangement is counted 3! times (number of ways to arrange the E's) in 7!.

Example 8: How many ways are there to arrange the letters in APPLE?

Solution: There are 5! ways if the letters are distinct, then we divide by 2! to account for the two P's, $\frac{5!}{2!} = \boxed{60}$.

If there are multiple sets of duplicate letters, then we divide to account for overcounting the same way.

Example 9: How many ways are there to arrange the letters in MISSISSIPPI?

Solution: There are 11 letters in MISSISSIPPI. There are 4 S's, 4 I's, and 2 P's. Therefore, the number of arrangements is $\frac{11!}{(4!)(4!)(2!)} = \boxed{165{,}765}$.

Example 10: How many ways are there to arrange the letters in SEEN if the two E's must stay together?

Solution: We can group the E's together and consider them as a "distinct letter": S(EE)N. Then there are $3! = \boxed{6}$ ways to arrange this.

Knowing how to deal with letter arrangements is important because many problems can be translated to the number of ways to arrange a word.

Example 11: Joyce has a collection of quarters, dimes, nickles, and pennies. If she has 5 quarters, 3 dimes, 4 nickles, and 6 pennies, how many ways are there to arrange the coins in a stack?

Solution: We can let quarters be represented by Q, dimes by D, nickles by N, and pennies by P. We want the number of arrangements of QQQQQDDDNNNNPPPPPP. This is $\frac{18!}{(5!)(3!)(4!)(6!)}$.

5 Combinations

Permutations are used when order matters. On the other hand, combinations are used when order doesn't matter. For instance, if we want to choose 4 people from a group of 8 to be part of a committee, we don't care about the order of the people. We only care about which of the 8 people are in the committee.

We denote the number of ways to choose r items out of n items as $\binom{n}{r}$, said n choose r. $\binom{n}{r}$ can be calculated using a simple formula.

Theorem 3 (Combination Formula)**.** The expression $\binom{n}{r}$ can be calculated as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Notice that this formula is very similar to the formula for evaluating a permutation. The only difference is that we're also dividing by $(n-r)!$.
To understand why this formula works, we can think about the arrangement of a word only two different letters. If we wanted to choose 4 people from a collection of 8 people, we can consider the arrangements of the "word" AAAABBBB. Each of the positions can be indexed to represent a distinct person.

1	2	3	4	5	6	7	8
A	A	A	A	B	B	B	B

Let the A's represent people (positions) that are chosen to be part of the 4-person group. While B's represent people that are not in the group. for instance, AABAABBB means that the people in positions 1,2,4, and 5 are chosen to be in the group. Then the number of arrangements of this word is the same as the number of ways to choose

4 people from 8 people. From the previous section, we know this is $\frac{8!}{(4!)(4!)}$, which is the same as the expression we'd get from using the formula. This example shows how the formula for $\binom{n}{r}$ is derived from considering the number of arrangements of a word with n instances of one letter and r instances of another letter.

Example 12: Compute $\binom{5}{2}$.

Solution:

$$\binom{5}{2} = \frac{5!}{(2!)(3!)} = \frac{120}{(2)(6)} = 10$$

Example 13: A club has 5 members from each of 3 different schools, for a total of 15 members. How many ways are there to arrange a presidency under the following conditions:
i. The club must choose one of the 3 schools at which to host the meeting, and

ii. The host school sends 2 representatives to the meeting, and each of the other two schools sends 1 representative. (Alcumus)

Solution: There are 3 ways to choose which school is the host school. There are $\binom{5}{2} = 10$ ways to choose the representatives for the host school. For each of the other schools, there are $\binom{5}{1} = 5$ ways to choose the representatives. This gives $(3)(10)(5)(5) = \boxed{750}$.

Next, we'll look at the three most popular counting techniques: complementary counting, constructive counting, and casework.

6 Complementary Counting

When you are asked to count something that is very complicated, you should consider counting the complement. Often the complement is easier to count.

Example 14: How many two-digit positive integers have at least one 7 as a digit? (AMC 10B 2004)

Solution: The complement is the set of numbers that have no 7's. Then the first digit has 8 choices (1,2,3,4,5,6,8,9) and the second digit has 9 choices (all digits besides 7). This gives $8 \cdot 9 = \boxed{72}$.

Example 15: How many integers in the set $25, 26, 27, \ldots, 250$ are not perfect cubes? (Alcumus)

Solution: We can count the number of perfect cubes in this interval and subtract it from the number of items in the interval $[25, 250]$. The first perfect cube in this interval is $3^3 = 27$. The last is $6^3 = 216$. Hence, there are $6 - 2 = 4$ perfect cubes between 25 and 250. There are $250 - 24 = 226$ positive integers in $[25, 250]$, so our answer is $226 - 4 = \boxed{222}$.

Observe how we counted the number of positive integers between 25 and 250, inclusive. We can consider the set of balls numbered 1 to 250. There are clearly 250 balls here. Then to get only the balls #25 to #250, we take away the first 24 balls (ball #1, ball #2, ..., ball #24). Hence, we want 250 - 24 = 226.

In general, we can use this method to find the number of elements in a list. The number of positive integers between a and b inclusive is $a-(b-1)=a-b+1$. This is because we can consider the set of a balls and subtract the first $b-1$ balls to get the number of balls between ball #b and ball #a.

Example 16: In a standard 52-card deck, how many 4-card hands are there that have at least one King?

Solution: The complement are the hands that have no Kings. This leaves 48 cards to choose from. Because order doesn't matter, there are $\binom{48}{4}$ hands that have no Kings. There are $\binom{52}{4}$ possible hands, so the number of hands with at least one King is $\binom{52}{4}-\binom{48}{4}=\boxed{76{,}145}$.

Seeing "at least one" in a problem is a hint that complementary counting might provide a simpler solution.

7 Constructive Counting

Sometimes you are asked to find the number of elements that have certain characteristics. Constructive counting is a technique where you construct the element with the specified characteristics, and that will reveal the number of elements.

Example 17: How many license plates can be formed if every license plate has 2 different letters (A through Z) followed by 2 different one digit $(0-9)$ numbers? (Alcumus)

Solution: There are 26 choices for the first letter and 25 choices for the second letter, since they must be distinct. Similarly, there are 10 choices for the first digit and 9 choices for the second digit. This gives $(26)(25)(10)(9)=\boxed{58{,}500}$

Example 18: The NHS executive team sat in a row during the induction dinner. There are 4 boys and 4 girls on the executive team. If no boy can sit next to another boy and no girl can sit next to another girl, how many ways are there for them to be seated.

Solution: There are only two ways the team can be seated so that the condition is filled. If G represents a boy and B represents a girl, these configurations are

$$BGBGBGBG \text{ and } GBGBGBGB$$

Within each arrangement above, there are 4! ways to arrange the boys and 4! ways to arrange the girls. This gives $(4!)(4!)+(4!)(4!)=16^2+16^2=\boxed{512}$.

Example 19: Ali, Bonnie, Carlo, and Dianna are going to drive together to a nearby theme park. The car they are using has four seats: one driver's seat, one front passenger seat, and two different back seats. Bonnie and Carlo are the only two who can drive the car. How many possible seating arrangements are there? (Alcumus)

Solution: There are two choices for the driver. After the driver is chosen, the remaining three people fill three other seats. This can be done in $3! = 6$ ways. Hence, there are $2(6) = \boxed{12}$ seating arrangements.

8 Casework

Casework is a counting technique where you break a problem into cases, then count the number of outcomes that satisfy each case. It is often the technique that is the most complicated, so you should consider if complementary or constructive counting can be used first. However, sometimes solving a problem by casework is necessary.

Example 20: A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen. (2013 AMC 10)

Solution: We can split the problem into two cases based on how many (1 or 2) math courses the student takes.

Case 1: The student takes both math classes.
This means 3 courses have already been chosen since English is always required. 3 courses remain to be the last class in the schedule. Each of these 3 remaining classes determines a unique schedule.

Case 2: The student takes exactly one math course.
The student has two choices for the math classes. After choosing the math class, 2 more classes must be chosen from History, Art, and Latin. This can be done in $\binom{3}{2} = 3$ ways. Then we multiply 2 because of the 2 ways to choose the math class. $2 \cdot 3 = 6$.

The total number of ways to make the program is the sum of our results from the two cases, $3 + 6 = 9$.

Example 21: The numbers 1447, 1005, and 1231 have something in common: each is a four-digit number beginning with 1 that has exactly two identity digits. How many such numbers are there? (Everaise)

Solution: There are two cases to consider: (1) 1 is the repeated digit, and (2) 1 is not repeated. The inspiration for these cases is that 1 is the digit that is always required.

Case 1 (1 is the repeated digit): There are 3 choices for where to put the other 1. Then there are $9 \cdot 8 = 72$ choices for the other two digits, since they must be distinct from 1 and each other. This gives $3 \cdot 72 = 216$.

Case 2 (1 is not repeated): We must choose two different digits, which can be done in 72 ways as above. If we let our our distinct digits be a and b, there are 3 ways to arrange them

$$1aab$$

$$1aba$$

$$1baa$$

This gives $3 \cdot 72 = 216$. Additionally, the observant reader will notice that we just arranged the word aab and there are $\binom{3}{2} = 3$ ways to do this.

We add our cases to get the answer $216 + 216 = \boxed{432}$

Example 22: How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits? (2005 AMC 10)

Solution: Let our number be represented by ABC, where A, B, C are the digits the three-digit number. First, observe that A and C must either both be even or both be odd. If their parity differs, then B won't be an integer. We could proceed with casework on the parity of A and C but that would become very ugly. Instead, notice that there are 9 choices for A (all digits except 0), and there are 5 choices for B once A is chosen. This is because once A is chosen, there are exactly 5 digits with the same parity. Also notice that each three-digit number is uniquely determined by the ordered pair (A, C). There are $9 \cdot 5 = \boxed{45}$ of these pairs.

9 Exercises

Problem 1: Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. How many license-plate patterns contain at least one palindrome? (2002 AIME 1)

Problem 2: Five men and nine women stand equally spaced around a circle in random order. What is the probability that every man stands diametrically opposite a woman? (2023 AIME 1).

Problem 3: How many 5-person committees can be selected from six teachers and eight students if there must be at least two students included? (CEMC)

Problem 4: How many positive integers less than 1000 have at least one odd digit? (CEMC)

Problem 5: At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts. (2021 AMC 10B)

Problem 6: How many positive 4-digit numbers begin with an odd digit or are divisible by 5? (CEMC)

Problem 7: A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to sreve on both committees. In how many different ways can a three-person planning committee be selected? (2013 AMC 10A)