DMMC Logarithms Handout

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1 Introduction

Many students, even those in high school, never have much exposure to logarithms. Because of this, they see them as complicated and possessing many rules. However, it's important to dedicate time to learning how to deal with logarithms due to their significance in the study of calculus, as well as in other disciplines. Additionally, their properties are quite intuitive and to be honest their just pretty fun to work with.

2 What are Logarithms?

There are two parts to a logarithmic expression: the **base** and the **argument**.

$$\log_b(x) = y \iff b \text{ raised to the } y^{th} \text{ power is } x$$

In the figure above, b is the base, x is the argument, and y is the output of the logarithmic function $\log_b(x)$. We can also see y as the value of $\log_b(x)$. In $\log_b(x)$, we are asking ”b raised to what power is equal to x?”. Whatever power answers the question is the value of $\log_b(x)$. In the general case above, it's y.

Example 1: What is the value of (a) $\log_2(4)$, (b) $\log_{10}(1000)$, and (c) $\log_2(1024)$

Solution: (a) The equivalent question is ”2 raised to what power equals 4?” $2^2 = 4$, so this power is 2 and $\log_2(4) = 2$. (b) The equivalent question is ”10 raised to what power equals 100?” $10^3 = 1000$, so this power is 3 and $\log_{10}(1000) = 3$. (c) The equivalent question is ”2 raised to what power equals 1024?” $2^{10} = 1024$, so this power is 10 and $\log_2(1024) = 10$.

Every valid logarithmic equation (i.e $\log_b(x) = y$) has an equivalent exponential equation, or exponential form. The logarithmic form and exponential form convey the same information but do so in different manners. As it's probably clear from their names, the logarithmic form uses logarithms and the exponential form uses exponents. It might not seem apparent now why these two forms matter but because they're equivalent, you can interchange between them, allowing you to use the properties of both logarithms and exponents.

Definition 1 (Logarithmic to Exponential Form)**.** *Given some equation in logarithmic form, we can convert to an equivalent equation in exponential form, and vice versa.*

$$\log_b(x) = y \iff b^y = x$$

Another way to affirm the above definition is to translate both of the equations to "words". Both translate to "b raised to the y^{th} power equals x".

Example 2: What is the exponential form equivalent to $\log_2(1024) = 10$?

Solution: Using Definition 1, we have

$$\log_2(1024) = 10 \iff 2^{10} = 1024$$

so the exponential form we want is $2^{10} = 1024$.

3 Domain and Range of Logarithmic Functions

You can think of $\log_b(x)$ as a function that outputs the power, y, b is raised to such that $b^y = x$. Observing the exponential form of a logarithmic equation reveals the domain and range of this function.

Although $\log_b(x)$ might be defined for some negative values of b, it is always assumed $b > 0$. Additionally, $b \neq 1$ since the equivalent exponential form is $1^y = x$, but if x is not -1 or 1, there is no value for y that satisfies the equation.

Consider $\log_b(x) = y$ and the corresponding exponential form $b^y = x$. For any real b and y, b^y cannot be less than or equal to 0. This means that x cannot be less than or equal to 0. We then have our domain because this is the only restriction on x.

Proposition 1. *The domain of* $\log_b(x)$ *is all real numbers greater than 0. It's domain is defined on* $(0, \infty)$

There isn't really any restriction on the output, y, of $\log_b(x)$. We can get 0 because $\log_b(1) = 0$. We can get negative real values when x is less than 1. For instance, $\log_2(1/2) = -1$. We can positive real values when x greater than 1. In particular, the output of $\log_b(x)$ is between 0 and 1 when $0 < x < b$

Proposition 2. *The range of* $\log_b(x)$ *is* $(-\infty, \infty)$.

4 Special Notation

When reading other math texts, you should be aware that it is common practice to write $\log_{10}(x)$ as $\log(x)$. The base being 10 is implied.

e is such an important number in mathematics that a logarithm with base e is given its own notation. That is, $\log_e(x)$ is equivalent to $\ln(x)$. But you will always see $\ln(x)$ used to represent the base e logarithm of x.

5 Logarithm Properties

Utilizing rules and properties of logarithms to manipulate equations involving them is where things get interesting.

Definition Property:

$$\log_b(b^k) = k$$

Proof. Instead of providing a rigorous proof of why this is true, it's more useful to know the intuition behind it (since then you can remember it better). By the definition, $\log_b(x)$ is the power, y, that b is to raised to such that $b^y = x$. then $\log_b(b^k)$ is the power b is raised to such that b raised to that power is equal to b^k. If we consider the equation $\log_b(b^k) = y$, its exponential form is $b^y = b^k$. The value of y that makes this equation true is $y = k$, so $\log_b(b^k) = k$. This is called the definition rule (only by me) because the it comes directly from the definition of a logarithm. □

Product Property:

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Proof. Let $m = \log_b(x)$ and $n = \log_b(y)$. Then from prior reasoning, we know $\log_b(x) = y \iff b^y = x$. This allows us to write

$$m = \log_b(x) \implies x = b^m$$

$$n = \log_b(y) \implies y = b^n$$

We care about the product xy since that is an argument of a logarithm in the declaration of the product rule, so we can multiply the two equations above.

$$xy = (b^m)(b^n) \implies xy = b^{m+n}$$

We can take the base b logarithm of both sides to get $\log_b(xy) = \log_b(b^{m+n} = m + n$, where the last equality comes from using the Definition Rule established above. Recall that at the beginning of the proof we let $m = \log_b(x)$ and $n = \log_b(y)$. If we substitute these back in we get $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$. □

Quotient Property:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Proof. Let $m = \log_b(x)$ and $n = \log_b(y)$. Then from prior reasoning, we know $\log_b(x) = y \iff b^y = x$. This allows us to write

$$m = \log_b(x) \implies x = b^m$$

$$n = \log_b(y) \implies y = b^n$$

Now we want x/y, so we can divide the first equation by the second equation.

$$\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n}$$

Once again, we use the Definition Property ($\log_b(b^k) = k$),

$$\log_b\left(\frac{x}{y}\right) = \log_b(b^{m-n}) \implies \log_b\left(\frac{x}{y}\right) = m - n$$

Finally, we substitute $m = \log_b(x)$ and $n = \log_b(y)$ to get

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

□

Observe how this proof is nearly identical to the proof for the Product Property. The only exception is we divide $x = b^m$ by $y = b^n$, where we multiply them in the other proof. But these moves are very motivated if we think about what we're trying to prove (we want a product so we multiply, or we want a fraction so we divide). This means that if you can recall the logic to prove one of these properties, you will also be able to recall the proof for the other property.

Power Property:

$$\log_b(x^k) = k \cdot \log_b(x)$$

Proof. Let $m = \log_b(x)$. In exponential form, we have $m = \log_b(x) \iff x = b^m$ To get the k^{th} power involved, which we want because it's in the statement of the Power Property, we raise both sides of the equation to the k power.

$$(x)^k = (b^m)^k \implies x^k = b^{mk}$$

We can then take the base b logarithm of both sides and use the Definition Property

$$\log_b(x^k) = \log_b(b^{mk}) \implies \log_b(m^k) = mk$$

Finally, we substitute $m = \log_b(x)$

$$\log_b(x^k) = k \cdot \log_b(x)$$

□

One-to-One Logarithmic Property:

$$\log_b(X) = \log_b(Y) \iff X = Y$$

The above property is one that we use in the previous proof to take the logarithm of both sides of an equation. You'll find it is essential to solving many problems involving logarithms.
There are two main ways it is used:
(1) You have an equation of the form $\log_b(X) = \log_b(Y)$, where it's very important that the bases are the same. Then from the one-to-one property, we have $X = Y$. This greatly simplifies the equation since now we don't have anymore logarithms.

(2) Often you want to have logarithms in an equation because of all the useful properties they have. Then if you have $X = Y$ (where X and/or Y are usually expressions with exponents), then you can take the base b (where b is some positive number chosen to make things simpler) logarithm of both sides of the equation due to the one-to-one property, resulting in $\log_b(X) = \log_b(Y)$.

Examples of how to use the one-to-one property in competition-style problems is provided in the Problem Solving chapter.

Base-Change Property:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Proof. Let $k = \log_a(x)$. In exponential form, $k = \log_a(x) \iff x = a^k$. We can then take the base b logarithm of both sides. It is important to mention that what b is doesn't matter, as long as it's greater than 0 and not equal to 1.

$$\log_b(x) = \log_b(a^k)$$

By the Power Rule,

$$\log_b(x) = k \cdot \log_b(a)$$

We can solve for k by dividing both sides by $\log_b(a)$. Then we can substitute $k = \log_a(x)$,

$$k = \frac{\log_b(x)}{\log_b(a)} \implies \log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

□

It is very important to know that it you can choose b when using the base-change formula (assuming you're going from $\log_a(x)$ to $\frac{\log_b(x)}{\log_b(a)}$). This is really useful because often you can choose a value for b that simplifies a problem.

Reciprocal Property:

$$\log_b(x) = \frac{1}{\log_x(b)}$$

Proof. We can utilize the base-change formula. Cleverly, we let the base of the logarithms in the fraction side of the equation be x.

$$\log_b(x) = \frac{\log_x(x)}{\log_x(b)} = \frac{1}{\log_x(b)}$$

where we used the fact that $\log_x(x) = 1$ since $x^1 = x$. □

6 Graphs

Knowing the characteristics of the graph of $y = \log_b(x)$ is important because you might be able to use $\log_b(x)$ to represent complex mathematical relationships. Its graph is the easiest way to identify if a certain event can be represented by a logarithmic model. All the features that are described can be found by analyzing $\log_b(x)$ with algebraic tools we've discussed so far.

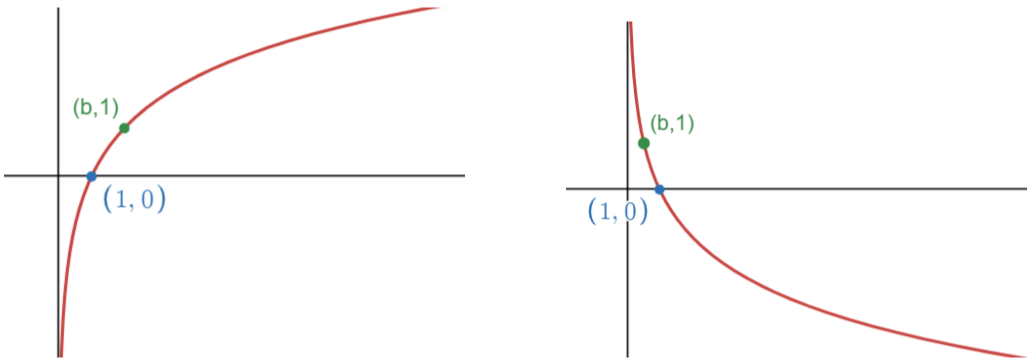

Figure 1: Graphs of $\log_b(x)$. On left, $b > 1$. On right, $0 < b < 1$

The reason two graphs are show in Figure 1 is because the shape of $\log_b(x)$ depends on whether $b > 1$ or $0 < b < 1$. Both have an x-intercept of (1,0). Recall that the x-intercept corresponds to $y = 0$. Then we have $\log_b(x) = 0$. This only occurs for $b = 1$. The point $(b, 1)$ is also important. It corresponds to $\log_b(b) = 1$, which is always true for valid b.

For $y = \log_b(x)$, the corresponding exponential form is $b^y = x$. For $b > 1$, as x closer and closer to 0, y gets larger in the negative direction, tending toward $-\infty$. This marks a vertical asymptote of $x = 0$. This occurs because for x to be smaller, y must be more negative (Recall that if a is positive, then $b^{-a} = 1/b^a$, as a increases, $-a$ becomes "more" negative, and $1/b^a$ decreases).
For $0 < b < 1$, the same asymptote is present, except y tends toward $+\infty$. This is because if we want x to be smaller, y must be greater since raising a number between 0 and 1 to a higher power results in a smaller output.
For $b > 1$, as x increases, y also increases since b must be raised to a greater exponent to result in b^y being greater.
For $0 < b < 1$, raising b to a more negative power results in b^y being larger.

This explanation of the relationship between x and y for the graph of $\log_b(x)$ might be a little confusing because we change between x changing (and the corresponding effect on y) and y changing (and the corresponding effect on x), but it's good to become proficient at determining the relationship between variables in an equation or function.

7 Problem Solving

Example: Find all solutions of $\log_2 x + \log_2(x - 4) = 5$ (UW-Plattville).

Solution: We can use the Product Property to combine the logarithms, $\log_2 x + \log_2(x - 4) = \log_2[x(x - 4)] = \log_2(x^2 - 4x) = 5$. Observe that we can represent 5 as a base 2 logarithm as $\log_2(32)$ since $2^5 = 32$. Then $\log_2(x^2 - 4x) = \log_2(32)$. By the One-to-One Property, $x^2 - 4x = 32 \implies x^2 - 4x - 32 = 0$. This factors as $(x - 8)(x + 4) = 0$, which produces solutions $x = 8$ and $x = -4$. However, $x = -4$ is not in the domain of $\log_2(x)$, so the only solution is $\boxed{x = 8}$.

Example: Solve for x: $\log_4 x - \log_4(x - 3) = \frac{1}{2}$ (UW-Plattville)

Solution: We can use the Quotient Property to combine the logarithms, $\log_4 x - \log_4(x - 3) = \log_4(\frac{x}{x-3}) = \frac{1}{2}$. $\frac{1}{2}$ can be represented as a base 4 logarithm as $\log_4 2$ since $4^{1/2} = 2$. Then $\log_4(\frac{x}{x-3}) = \log_4 2$. By the One-to-One Property, $\frac{x}{x-3} = 2$. We can multiply both sides by $x - 3$ and solve for x: $x = 2(x - 3) = 2x - 6 \implies \boxed{x = 6}$.

Example: Solve for all values of x that satisfy $(\log_2 x)^2 + \log_2(x^3) + 2 = 0$.

Solution: We can use the Power Property to simplify $\log_2(x^3) = 3\log_2(x)$. The equation becomes $(\log_2 x)^2 + 3\log_2 x + 2 = 0$. This is an equation in terms of $\log_2 x$, so we can let $a = \log_2 x$ and solve the quadratic $a^2 + 3a + 2 = 0$. It factors as $(a+2)(a+1)$, which means $a = -2$ and $a = -1$. If $a = -2$, then $\log_2 x = -2 \implies x = 2^{-2} = 1/4$. If $a = -1$, then $\log_2 x = -1 \implies x = 2^{-1} = 1/2$. The solutions are $\boxed{x = 1/4, 1/2}$.

Example: If $\log_a 2 = y, \log_a 5x = 105$ and $\log_a 2x = 35$, compute $\log_a 10$. (UW-Platteville)

Solution: We can subtract the second logarithm from the third logarithm and use the Quotient property to combine them. We choose this combination because then the x will be eliminated; we don't want it since it's not in the expression we want to find the value of. $\log_a(5/2) = \log_a 5x - \log_a 2x = 105 - 35 = 70$. By the product rule, we can add $\log_a 4$ to $\log_a(5/2)$ to get $\log_a 10$. By the Power Rule, $\log_a 4 = \log_a(2^2) = 2\log_a(2) = 2y$. Then $\log_a 10 = \log_a(5/2) + \log_a(4) = \boxed{\text{70 + 2y}}$

Example: There exist real numbers x and y, both greater than 1, such that $\log_x(y^x) = \log_y(x^{4y}) = 10$. Find xy. (AIME)

Solution: By the Power Property, we have $x\log_x y = 4y\log_y x = 10$. This can be broken up into two separate equations:

$$x\log_x y = 10$$

$$4y\log_y x = 10$$

We multiply the equations:

$$4xy(\log_x y \cdot \log_y x) = 100$$

By the Base-Change Property, $\log_x y = \frac{\log y}{\log x}$ and $\log_y x = \frac{\log x}{\log y}$. Therefore, $\log_x y \cdot \log_y x = \frac{\log y}{\log x} \cdot \frac{\log x}{\log y} = 1$. The equation becomes

$$4xy(\log_x y \cdot \log_y x) = 4xy = 100$$

By dividing 4 by both sides, $\boxed{xy = 25}$

8 Summary

- Every logarithmic expression has a base and an argument. $\log_b(x)$ can be defined as
$$\log_b(x) = y \iff b \text{ raised to the } y^{th} \text{ power is } x$$
- Converting from logarithmic to exponential form is an important problem-solving tool,
$$\log_b(x) = y \iff b^y = x$$
- The domain of $\log_b(x)$ is $(0, \infty)$.
- The range of $\log_b(x)$ is $(-\infty, \infty)$.
- $\log(x)$ is shorthand for $\log_{10}(x)$, and $\ln(x)$ denotes $\log_e(x)$.

- Logarithms have numerous properties that are useful for solving problems:
  - Definition Property: $\log_b(b^k) = k$
  - Product Property: $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
  - Quotient Property: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
  - Power Property: $\log_b(x^k) = k \cdot \log_b(x)$
  - One-to-One Property: $\log_b(X) = \log_b(Y) \iff X = Y$
  - Base-Change Property: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
  - Reciprocal Property: $\log_b(x) = \frac{1}{\log_x(b)}$
- Knowing the features of the graph $y = \log_b(x)$ can be useful when drawing connections with mathematical relationships.

9 Problems

After going through the material provided in this document, please try some of these problems that we collected. Do not be discouraged by their difficulty, nearly all of them come from challenging national competitions. However, if you dedicate the time to really trying to puzzle them out, you skills with logarithms will improve much more than if you did twice as many simple problems.
Under each problem is the competition that it came from. If you come to a answer for a specific problem, you can look up the name of the contest and find that problem with its solution. Even if you got the answer correct, it's still helpful to go through the solution (often there are multiple solutions that have different underlying principles). If you get stuck on a problem, consider coming back to it the next, then if you're really stuck look at the solution. But don't just gloss over it and more onto the next problem. Really try to understand it so you can apply that understanding to future problems.
Lastly, some problems are ones that we came up with (the ones without a test name). If you want to know the answer, please email us at desmoinesmathcircle@gmail.com. You can also email us if there's a solution you're having trouble with and we'll do our best to get back to you with an explanation.

Problem 1: Find the value of

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \lfloor \log_2 4 \rfloor + \lfloor \log_2 5 \rfloor + \lfloor \log_2 6 \rfloor + \lfloor \log_2 7 \rfloor + \lfloor \log_2 8 \rfloor$$

where for real x, $\lfloor x \rfloor$ denotes the greatest integer less than x.

Problem 2: Find all solutions of $\log_3 x - \log_3(x - 4) = 2$.

Problem 3: Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $\left(\log_2 \frac{x}{y}\right)^2$?
(2019 AMC 12A)

Problem 4: Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.
(1984 AIME)

Problem 5: Let x, y, and z all exceed 1 and let w be a positive number such that $\log_x w = 24, \log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.
(1983 AIME)

Problem 6: What is the value of a for which

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$$

(2015 AMC 12A)

Problem 7: For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 1000$ and that $\lfloor \log_2 n \rfloor$ is a positive even integer?
(1996 AIME)

Problem 8: What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27$$

(2018 AMC 12B)

Problem 9: Positive real numbers $b \neq 1$ and n satisfy the equations

$$\sqrt{\log_b n} = \log_b \sqrt{n}$$

$$b \cdot \log_b n = \log_b(bn)$$

The value of n is $\frac{j}{k}$, where j and k are relatively prime positive integers. Find $j + k$.
(2023 AIME I)

Problem 10: There is a unique positive integer n such that $\log_2(\log_{16} n) = \log_4(\log_4 n)$. What is the sum of the digits of n?
(2020 AMC 12A)

Problem 11: For how many positive integers x is $\log_{10}(x - 40) + \log_{10}(60 - x) < 2$?
(2014 AMC 12B)

Problem 12: Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.
(1988 AIME)