Omega Sample Test 1 Solutions

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Answer Key

1. 16

2. 100°

3. 66

4. 7/4

5. 792

6. $\sqrt{\frac{40+6\sqrt{3}}{4}}$

Solutions

1. What is the sum of the coefficients of the polynomial $(x+y)^4$?

Solution 1: Recall that when we want to find the sum of the coefficients for some polynomial $f(x)$ in x, we let $x = 1$ and find $f(1)$. Similarly for a polynomial $f(x,y)$ in x and y, we let $x = 1$ and $y = 1$. We find $f(1,1)$. $(1+1)^4 = 2^4 = 16$, so the sum of the coefficients is 16.

Solution 2: We could expand out $(x + y)^4$ using the binomial theorem. By doing this we would see that the coefficients follow the 4th row of Pascal's triangle. In general, the coefficients of $(x+y)^n$ can be derived from the elements in the nth row of Pascal's. Furthermore, the sum of the elements in the nth row of Pascal's triangle is 2^n. Because the coefficients of $(x+y)^4$ are the elements in the 4th row of Pascal's triangle, we want the sum of these elements. This is $2^4 = 16$.

2. A circle has an inscribed angle with measure 50°. What is the measure of the arc the inscribed angle intersects?

Solution: An inscribed angle is half the measure of the arc it subtends. Therefore, the arc is twice the inscribed angle, making the arc have a measure of 100°.

3. A tennis tournament is structured so that every player in the tournament plays every other player exactly once. If the tournament has 12 players, how many games are played?

Solution 1: Each of the 12 people can play a game with each of the 11 other people. This gives $11(12) = 132$. However, this overcounts games. Consider two people, Person 1 and Person 2, that play a game against each other. In this count, we count the game for Person 1 vs. Person 2 and the game for Person 2 vs. Person 1. Therefore, our count of 132, overcounts the number of games by 2, so we divide by 2. $132/2 = 66$ games.

Solution 2: $\binom{12}{2}$ represents the number of ways to construct groups of 2 (without regard to order) from 12 distinct items, which is exactly what we want to count since each group represents a game between two players. The formula for evaluating binomial coefficients is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

With $n = 12$ and $k = 2$, we have

$$\binom{12}{2} = \frac{12!}{2!(12-2)!} = \frac{12!}{2(10!)} = \frac{12 \cdot 11}{2} = \frac{132}{2} = 66$$

4. What is the least value of the function $f(x) = x^2 + 3x + 4$?

Solution: There are numerous solutions to this such as using calculus or finding the vertex of the parabola $x^2 + 3x + 4$. However, we'll find the minimum by completing the square. The perfect square trinomial from $x^2 + 3x$ is $(x + \frac{3}{2})^2$ since this will have a term of $3x$ when expanded. To allow this factorization, we must add $\frac{9}{4}$ to $x^2 + 3x + 4$.
To do this, we can cleverly add 0. That is, we'll add $\frac{9}{4}$ and subtract $\frac{9}{4}$ from $x^2 + 3x + 4$. This makes it so equality is still maintained. $x^2 + 3x + 4 = x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 4 = (x + \frac{3}{2})^2 - \frac{9}{4} + 4$. Notice that we used the factorization $x^2 + 3x + \frac{9}{4} = (x + \frac{3}{2})^2$. Then we have $(x + \frac{3}{2})^2 - \frac{9}{4} + 4 = (x + \frac{3}{2})^2 + \frac{7}{4}$.
One can surmise that the minimum value of $(x + \frac{3}{2})^2$ is 0 (which occurs at $x = -3/2$ since there is no way for the square of a number to be negative. This concept is called the trivial inequality! Then if $(x + \frac{3}{2})^2 = 0$, all that is left is the $7/4$, which is constant so there is no way to get any lower. $7/4$ is the function's minimum value.

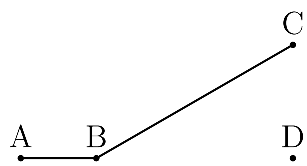
5. A caterpillar is at (0,0) on the xy-plane. How many are ways are there for the caterpillar to get to the point (5,7) by only making moves of going up or right?

Solution: Observe that any way of getting to (5,7) from (0,0) corresponds to an arrangement of RRRRRUUUUUUU, where there are 5 R's and 7 U's. Each R corresponds to moving right by one and each U corresponds to moving up by one. This is because you can simplify any way of going from (0,0) to (5,7) to permutation of rights and lefts.
We are left with determining the number of arrangements of RRRRRUUUUUUU. This is $\binom{12}{5}$. One way to think of this is that you have 12 different

spaces for letters (since there are 12 letters in total) and you choose 5 spaces for the R's to go. Then the U's just fill in the remaining spaces. You can calculate $\binom{12}{5} = 792$ using the formula for calculating binomial coefficients shown in Solution 2 of Problem 3.

6. Martha is going on a run. She begins by running 1 mile east. Then she turns and runs 30° north of the x-axis for 3 miles. Then she stops to get a drink of water. How far is she away from her starting point?

Solution: We can begin by drawing a diagram,



where AB is the 1 mile long segment and BC is the 3 mile long segment. Then the distance we want is AC. It can be found by dropping the perpendicular from C to D and considering the right triangle ACD. If we can find AD and CD, we can use the pythagoren theorem to find AC.
Because $\angle CBD$ is 30°, we can use 30-60-90 triangle properties. If $BC = 3$, then $CD = 3/2$ and $BD = \frac{3}{2}\sqrt{3}$. These might seem to come out of nowhere, but hopefully the DMMC will soon have a handout on special triangles and special triangles will become a breeze to you! Then we know $AD = AB + BD = 1 + \frac{3}{2}\sqrt{3} = \frac{2+3\sqrt{3}}{2}$. We can now use the Pythagorean Theorem with AD $= \frac{2+3\sqrt{3}}{2}$ and $CD = \frac{3}{2}$.

$$AC^2 = \left(\frac{2+3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{4+6\sqrt{3}+27}{4} + \frac{9}{4} = \frac{40+6\sqrt{3}}{4}.$$

Taking the square root of both sides,

$$AC = \sqrt{\frac{40+6\sqrt{3}}{4}}$$