Omega Sample Test 2 Solutions

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Answer Key

1. 16

2. 20

3. 1815

4. $\frac{4}{1+\sqrt{3}}$

5. 36

6. 11

Solutions

1. $\lfloor x \rfloor$ denotes the greatest integer less than x, for all real numbers x. For example, $\lfloor 5.4 \rfloor = 5$. Compute

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \lfloor \sqrt{4} \rfloor + \lfloor \sqrt{5} \rfloor + \lfloor \sqrt{6} \rfloor + \lfloor \sqrt{7} \rfloor + \lfloor \sqrt{8} \rfloor + \lfloor \sqrt{9} \rfloor$$

Solution: Observe that $\lfloor \sqrt{x} \rfloor$, where x is a perfect square, is an integer. So $\lfloor \sqrt{x} \rfloor = \sqrt{x}$. Then all $\lfloor \sqrt{a} \rfloor$ between $\lfloor \sqrt{x} \rfloor$ and the next perfect square term evaluate to $\sqrt{x}$. For instance, $\lfloor \sqrt{1} \rfloor = 1$ and $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \sqrt{3} \rfloor = 1$. But then $\lfloor \sqrt{4} \rfloor = 2$.
Using this observation, we can quickly calculate each of the terms: $\lfloor \sqrt{1} \rfloor = 1, \lfloor \sqrt{2} \rfloor = 1, \lfloor \sqrt{2} \rfloor = 1, \lfloor \sqrt{4} \rfloor = 2, \lfloor \sqrt{5} \rfloor = 2, \lfloor \sqrt{6} \rfloor = 2 \lfloor \sqrt{7} \rfloor = 2, \lfloor \sqrt{8} \rfloor = 2, \lfloor \sqrt{9} \rfloor = 3$.
Then $1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 3 = 16$.

2. How many positive divisors does $2^3 \cdot 3^4$ have?

Solution: Suppose we have some positive integer n with prime factorization $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_n^{e_n}$.
Then the number of divisors of n is given by

$$(e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$$

You can construct this from combinatorics argument. Any integer of the form $p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_3}$, where a_i is less than or equal to the corresponding power that p_i is raised to in the actual prime factorization, is a divisor of n.
For instance, $2^2 \cdot 3^3$ is divisor of $2^3 \cdot 3^4$ but 2^5 is not. Then for $2^3 \cdot 3^4$, thee number of divisors is $(3 + 1)(4 + 1) = 4(5) = 20$.

3. What is the sum of the positive divisors of $2^3 \cdot 3^4$

Solution: To get a sum of all the positive divisors, we want to want multiply the sums of powers of each prime. This is probably easier to understand when you see the actual product. For $2^3 \cdot 3^4$ the sum of divisors is given by $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2 + 3^3 + 3^4)$. If you visualize the expansion of the product, you can see how this gives every divisor of $2^3 \cdot 3^4$. Each of the geometric series can be evaluated using the formula

$$S = \frac{a(r^n - 1)}{r - 1}$$

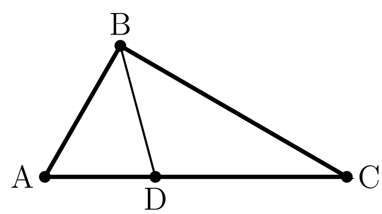
where r is the common ratio, a is the first term, and n is the number of terms.
Using this formula,

$$1 + 2^1 + 2^2 + 2^3 = \frac{1(2^4 - 1)}{2 - 1} = \frac{15}{1} = 15$$

$$1 + 3^1 + 3^2 + 3^3 + 3^4 = \frac{1(3^5 - 1)}{3 - 1} = \frac{242}{2} = 121$$

Therefore, the sum of the divisors of $2^3 \cdot 3^4$ is $15(121) = 1815$.

4. ABC is a right triangle with $AB = 2$, $BC = 2\sqrt{3}$, and $AC = 4$. If BD is the angle bisector of $\angle ABC$, what are the lengths of AD and DC?



Solution: There is an interesting theorem that can be used when you have the angle bisector of triangle. It is justly called the Angle Bisector Theorem: The angle bisector divides the side opposite to it into two parts proportional to the other two sides. For $\triangle ABC$ this means

$$\frac{AD}{DC} = \frac{AB}{BC}$$

$$\frac{AD}{DC} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \implies DC = \sqrt{3}AD$$

However we're also give $AC = 4$ and $AC = AD + DC \implies AD + DC = 4$. We can substitute $DC = \sqrt{3}AD$: $AD + \sqrt{3}AD = (1 + \sqrt{3})AD = 4 \implies AD = \frac{4}{1+\sqrt{3}}$.

5. Kevin wants to buy 7 new shirts. At the department store, they have 3 colors to chose from: blue, pink, and gray. How many ways are there for Kevin to choose the colors for his 7 shirts?

Solution: We can see the 7 shirts, before colors are assigned to them, as being indistinguishable from each other. Because we have indistinguishable objects that we want to put in distinct groups, we can use stars and bars. Stars and bars is a technique that is used to find the number of ways to put n indistinguishable objects into k distinguishable groups.

For this problem, suppose we have 7 stars (X). These represent the shirts. Then we also have two bars (|). These represent the groups we're putting the shirts in, which are based on color. But notice that we have one less bar than the number of color groups we have. This is because we only need two bars to determine three groups. This can be seen in as XX|XXX|XX.
This arrangement of the X's and |'s symbolize a way of grouping the shirts. Suppose we let the X's to the left of the | be blue shirts, the X's in between the |'s are pink shirts, and the X's to the right of the second | are gray shirts.

So with any arrangement of 7 X's and 2 |'s, we get a successful grouping of the shirts into the color groups. So now we're just curious how many ways are there to arrange 7 X's and 2 |'s. This can be done in $\binom{9}{2}$ since we have 9 places for items go and we can choose 2 places for the |'s to go. Then the X's just fill in the remaining places. Using the binomial coefficient formula, we can calculate $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$

6. Joseph created a new money system. The only coins are 3-cent coins and 7 cent-coins. What is the largest number of cents Joseph cannot create with his money system?

Solution: There exists a theorem called the Chicken McNugget Theorem. It states that if we have two positive integers m and n that are relatively prime (they share no divisors greater than 1) then the greatest positive integer that cannot be written as a sum of a multiple of m and a multiple of n is $mn - m - n$.
Utilizing this theorem, we can conclude that the greatest number of cents that cannot be made up of 3-cent coins and 7-cent coins is $3(7) - 3 - 7 = 11$ cents.