Omega Sample Test 3 Solutions

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Answer Key

1. 33

2. 2

3. 35

4. 2^{81}

5. 7

6. 4

Solutions

1. How many positive integers less than 100 have a sum of digits divisible by 3?

Solution: This problem is simpler than it seems because a number is divisible by 3 if and only if the sum of the digits of the number is divisible by 3. So we can just find the number of positive integers less than 100 that are divisible by 3. This is $\lfloor \frac{100}{3} \rfloor = 33$. Or you could recognize that the smallest positive integer divisible by 3 is $3 \cdot 1$ and the largest is $3 \cdot 33$, so you get all the multiples between them (inclusive), which is 33.

2. What is the remainder when 2^{101} is divided by 3?

Solution 1: When working with some modulus m, if $a \equiv b$ (mod m), then

$$a^n \equiv b^n \text{ (mod m)}$$

Observe that $2 \equiv -1$(mod 3). Using the property of modular arithmetic shown above, we have $2^{101} \equiv (-1)^{101}$ (mod 3). And -1 raised to any odd power is -1 so we have 2^{101} leaves a remainder of $-1 \equiv 2$ when divided by 3.

Solution 2: Observe that $2^3 = 8 \equiv -1$ (mod 3). Because $101 = 3 \cdot 33 + 2$, we can write $2^{101} = (2^3)^{33} \cdot 2^2 \equiv (-1)^{33} \cdot 2^2$ (mod 3). Then $(-1)^{33} \cdot 2^2 = -4 \equiv 2$ (mod 3).

3. A donut shop has donuts with sprinkles, frosting, or both. 20 of the donuts have sprinkles, 30 have frosting, and 15 have both. How many donuts does the shop have?

Solution: We can use the Principal of Inclusion-Exclusion. For two events, A and B, it states

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$$

We can explain a lot of this unfamiliar notation. $\#(X)$ is the number of elements in some set that fulfill the condition X. $X \cap Y$ are the elements for which both event X and event Y occur. An example of this is when X is divisible by 2 and Y is when a number is divisible by 3. Then a number in $X \cap Y$ is 6 since 6 is divisible by 2 and 3. $X \cup Y$ are all the elements in a set that fulfill X or fulfill Y, or that are in either set X or set Y. For instance, if $X = \{1,2,3\}$ and $Y = \{3,4,5\}$, then $X \cup Y = \{1,2,3,4,5\}$. On the other hand, $X \cap Y \{3\}$ since the only common element between the sets is 3. Then the PIE (Principle of Inclusion-Exclusion) for this problem is Let our events be A = donut has sprinkles and B = donut has frosting. Having both sprinkles and frosting isn't an event because it is the same as $A \cap B$. Notice that from PIE we're calculating $A \cup B$, which is the number of donuts that have sprinkles or frosting. But why is this is the same as the number of total donuts? This is because in the problem statement we know that donuts have either frosting, sprinkles, or both. There is no donut that doesn't have frosting or sprinkles. So counting the number of donuts with frosting or sprinkles counts the number of total donuts.

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B) \implies \#(A \cup B) = 20 + 30 - 15 = 35$$

Therefore, the shop has a total of 35 donuts.

4. Solve for x:

$$\log_4(\log_3(\log_2(x))) = 1$$

Solution: This is a very complicated equation and we're unfamiliar with taking the logarithm of expressions that involving logarithms. So let $y = \log_3(\log_2(x))$. Then we want $\log_4(y) = 1$. It's clear that $y = 4$ since 1 is the power 4 is raised to to equal 4. Then we have

$$y = \log_3(\log_2(x)) = 4$$

Once again, we let $\log_2(x) = z$. $\log_3(z) = 4 \implies z = 3^4 = 81$. We are left with $\log_2(x) = 81 \implies x = 2^{81}$.

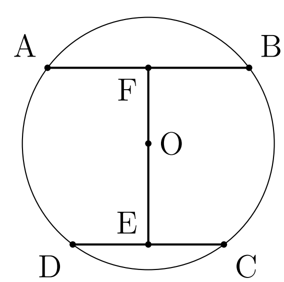
This problem demonstrates the importance of making substitutions when you're faced with a complicated problem. We're not familiar with taking the logarithm of a more logarithms, so we let those logarithm arguments be equal to some variable. This significantly simplified the problem because we were able to think about it the way we normally think of logarithm problems.

5. Alice is hosting a Super Bowl party. She doesn't know how many people

will come, but that at least 1 person will attend, and at most 5 people. If she has 5 cans of soda, how many ways are there for her to distribute the cans for any amount of people coming? For instance, if three people attend, she can distribute them 3-1-1. If four people attend, she can distribute them 2-2-1-0. Assume that the people are indistinguishable and she isn't including herself.

Solution: This is a partitions problems. In partition problems, we want to find the number of ways to divide some number of indistinguishable items into indistinguishable groups.
There are many ways to solve this problem but the method I would like to implement is creating cases based on the number of people that go to the party (there could be 1, 2, 3, 4, or 5 people at the party) then we count the successful distributions of 5 (since there are 5 cans of soda) for that number of people.
Case 1: 1 Person - Then only partition is 5, so this case gives 1 partition of 5.
Case 2: 2 People - The partitions are 1-4, 2,3. This case gives 2 partitions.
Case 3: 3 People - The partitions are 3-1-1 and 2-2-1. This case gives three partitions.
Case 4: 4 People - The only partition is 2-1-1-1.
Case 5: 5 People - The only partition is 1-1-1-1-1. If we sum the number of partitions over the 5 cases, we get 7. This means there are 7 ways to distribute 5 cans of soda to at most 5 people.

6. Circle O has radius 5. Two chords, AB and AC, are drawn parallel to each other. If EC = 3 and the distance between the two chords is 7, then what is the length of FB?



Solution: We can first draw the line segment from O to C. OC has length 5 since it is a radius of the circle O. We know $EC = 3$, so by Pythagorean

Theorem on $\triangle OEC$, $OE = 4$. If we are told the distance between the two chords is 7, then $FE = 7$. If $FE = 7$ and $OE = 4$, we have $FO = FE - OE = 7 - 4 = 3$. Once again we can draw radius OB to get right triangle BFO. We have $FO = 3$ and $BO = 5$, so by Pythagorean theorem $BF = 4$.